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Equivalent Lagrangians in field theory†

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Abstract. A theorem by Hojman and Harleston for multidimensional equivalent Lagrangians (L and $\bar{L}$) in the realm of Newtonian mechanics is generalised to field theory. Equivalent Lagrangians for some set of field equations are found.

1. Introduction

Canonical quantisation is based on the existence of a Lagrangian function. If, given a set of dynamical equations, in correspondence we have two (or more) Lagrangian functions, the theory, in principle, can admit different quantisation procedures. This problem motivated a line of research (the search for equivalent Lagrangians) in the realm of classical mechanics (Okubo 1980 and references therein, Hojman and Harleston 1981). Two Lagrangians are said to be equivalent iff they lead non-trivially to the same equations.

The mechanical counterpart for a classical discrete system is achieved if we consider the Lagrangian $L = L(q^i, \dot{q}^i, t)$ that leads to the set of n Euler–Lagrange equations

$$G_i(q, \dot{q}, \ddot{q}, t) = (\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0 \qquad i = 1, 2, \ldots, n. \tag{1}$$

A set of equations equivalent to equation (1) may be constructed with the aid of a non-singular matrix $\Delta^n_m(q^i, \dot{q}^i, t)$

$$\bar{G}_i(q, \dot{q}, \ddot{q}, t) = (\mathrm{d}/\mathrm{d}t)(\partial \bar{L}/\partial \dot{q}^i) - \partial \bar{L}/\partial q^i = 0 \tag{2}$$

provided $\bar{G}_i = \Delta_i^k G_k$ (the summation convention is used). If equations (1) and (2) are valid, we say that L and $\bar{L}$ are equivalent.

The natural generalisation of this problem in field theory is

$$\bar{F}_a(\phi^c, \phi^{c;}{}_{\mu}, \phi^{c;}{}_{\mu\nu}, x^\alpha) = h_a{}^b(\phi^c, \phi^{c;}{}_{\mu}, x^\nu) F_b(\phi^c, \phi^{c;}{}_{\mu}, \phi^{c;}{}_{\mu\nu}, x^\nu) \tag{3}$$

such that

$$F_a(\phi^c, \phi^{c;}{}_{\mu}, \phi^{c;}{}_{\mu\nu}, x^\alpha) = (\mathrm{d}/\mathrm{d}x^\mu)(\partial \mathscr{L}/\partial \phi^{a;}{}_{\mu}) - \partial \mathscr{L}/\partial \phi^a = 0$$

and

$$\bar{F}_a(\phi^c, \phi^{c;}{}_{\mu}, \phi^{c;}{}_{\mu\nu}, x^\alpha) = (\mathrm{d}/\mathrm{d}x^\mu)(\partial \bar{\mathscr{L}}/\partial \phi^{a;}{}_{\mu}) - \partial \bar{\mathscr{L}}/\partial \phi^a = 0$$

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where

$$\phi^{c;}{}_{\mu}=\partial\phi^{c}/\partial x^{\mu} \qquad \phi^{c;}{}_{\mu\nu}=\partial^2\phi^{c}/\partial x^{\mu}\,\partial x^{\nu}$$
$$a,b,c=1,2,\ldots,n \qquad \alpha,\mu,\nu=0,1,2,3.$$

$\mathscr{L}$ and $\bar{\mathscr{L}}$ are said to be equivalent Lagrangian densities.

Under the above conditions, it was proved by Hojman and Harleston (1981) that

$$\mathrm{d}(\Delta^k_k)/\mathrm{d}t=0 \qquad \text{i.e. } \mathrm{d}[\mathrm{Tr}(\Delta)]/\mathrm{d}t=0.$$

This result was also derived by Henneaux (1981) and Lutzky (1982).

Here, we generalise this theorem for fields of the type

$$F_a=g^{\mu\nu}A_{ab}(x_\alpha,\phi^c,\phi^{c;}{}_{\alpha})\phi^{b;}{}_{\mu\nu}+B_a(x_\alpha,\phi^c,\phi^{c;}{}_{\alpha})=0 \tag{4}$$

where $g^{00}=-g^{kk}=1$, $g^{\mu\nu}=0$, $\mu\neq\nu$, $k=1,2,3$; $A_{ab}=A_{ba}$, $\det(A)\neq 0$. Most field theories are included in this category.

In particular, we will prove the following theorem.

Theorem. If $\bar{\mathscr{L}}$ is equivalent to $\mathscr{L}$ then

$$\mathrm{d}(h_a{}^a)/\mathrm{d}x^{\mu}=0 \qquad \text{i.e. } \mathrm{d}[\mathrm{Tr}(h)]/\mathrm{d}x^{\mu}=0 \tag{5}$$
$$a=1,2,\ldots,n \qquad \mu=0,1,2,3.$$

In order to derive this result we use Santilli's (1977) conditions.

2. Santilli conditions

Santilli conditions are necessary and sufficient conditions for a given set of field equations $F_a(x^\alpha,\phi^c,\phi^{c;}{}_{\alpha},\phi^{c;}{}_{\alpha\beta})=0$, $a=1,2,\ldots,n$; $\alpha,\beta=0,1,2,3$, to be derived from Hamilton's variational principle. They are

$$\frac{\partial F_a}{\partial\phi^{b;}{}_{\mu\nu}}=\frac{\partial F_b}{\partial\phi^{a;}{}_{\nu\mu}}=\frac{\partial F_a}{\partial\phi^{b;}{}_{\nu\mu}}=\frac{\partial F_b}{\partial\phi^{a;}{}_{\mu\nu}} \tag{6a}$$

$$\frac{\partial F_a}{\partial\phi^{b;}{}_{\mu}}+\frac{\partial F_b}{\partial\phi^{a;}{}_{\mu}}=2\frac{\mathrm{d}}{\mathrm{d}x^{\nu}}\left(\frac{\partial F_a}{\partial\phi^{b;}{}_{\mu\nu}}\right) \tag{6b}$$

$$\frac{\partial F_a}{\partial\phi^{b}}-\frac{\partial F_b}{\partial\phi^{a}}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x^{\mu}}\left(\frac{\partial F_a}{\partial\phi^{b;}{}_{\mu}}-\frac{\partial F_b}{\partial\phi^{a;}{}_{\mu}}\right) \tag{6c}$$

$$a,b=1,2,\ldots,n \qquad \mu,\nu=0,1,2,3.$$

It is known that the Euler–Lagrange equations for continuous systems are linear in the second-order partial derivatives $\phi^{c;}{}_{\mu\nu}$. This is seen immediately from equations (6*a*) and (6*b*). Hence, without loss of generality we can restrict ourselves to equations of the form

$$F_a=A^{\mu\nu}_{ab}(x_\alpha,\phi^c,\phi^{c;}{}_{\alpha})\phi^{b;}{}_{\mu\nu}+B_a(x_\alpha,\phi^c,\phi^{c;}{}_{\alpha})=0 \tag{7}$$
$$a,b,c=1,2,\ldots,n \qquad \mu,\nu,\alpha=0,1,2,3 \qquad A^{\mu\nu}_{ab}=A^{\mu\nu}_{ba}.$$

Using this result, the Santilli conditions are transformed into

$$A_{ab}^{\mu\nu} = A_{ba}^{\nu\mu} = A_{ab}^{\nu\mu} \tag{8a}$$

$$A_{ac}^{\nu\alpha;\mu}{}_{b} + A_{bc}^{\nu\alpha;\mu}{}_{a} = A_{ba}^{\mu\nu;\alpha}{}_{c} \tag{8b}$$

$$A_{ad}^{\alpha\beta;\mu;\nu}{}_{b\ c} = A_{bd}^{\alpha\beta;\mu;\nu}{}_{a\ c} \tag{8c}$$

$$B_a{}^{;\mu}{}_b + B_b{}^{;\mu}{}_a = 2(\partial/\partial x^\nu + \phi^{c;}{}_\nu\, \partial/\partial\phi^c)A_{ab}^{\mu\nu} \tag{8d}$$

$$B_a{}^{;}{}_b - B_b{}^{;}{}_a = \tfrac{1}{2}(\partial/\partial x^\nu + \phi^{c;}{}_\nu\, \partial/\partial\phi^c)(B_a{}^{;\nu}{}_b - B_b{}^{;\nu}{}_a) \tag{8e}$$

where

$$A_{ab}^{\mu\nu;\alpha}{}_{c} = \partial A_{ab}^{\mu\nu}/\partial\phi^{c;}{}_\alpha \qquad B_a{}^{;}{}_b = \partial B_a/\partial\phi^b \qquad \text{etc}$$

$$A_{ab}^{\mu\nu;\alpha}{}_{c} = A_{ab}^{\mu\nu;\alpha}{}_{c} + A_{ab}^{\mu\alpha;\nu}{}_{c} \qquad A_{ab}^{\mu\nu;\alpha;\beta}{}_{c\ d} = A_{ab}^{\mu\nu;\alpha;\beta}{}_{c\ d} + A_{ad}^{\mu\nu;\alpha;\beta}{}_{c\ b}$$

$$a, b, c, d = 1, 2, \ldots, n \qquad \mu, \nu, \alpha, \beta = 0, 1, 2, 3.$$

The existence of $\bar{\mathscr{L}}$ will be ensured by imposing the above conditions on $\bar{F}$ which yields (using equation (3))

$$h_a{}^c A_{cb}^{\mu\nu} = h_b^c A_{ca}^{\nu\mu} = h_a^c A_{cb}^{\nu\mu} \tag{9a}$$

$$(h_a{}^d A_{dc}^{\nu\alpha})^{;\mu}{}_b + (h_b{}^d A_{dc})^{;\mu}{}_a = (h_b{}^d A_{da}^{\mu\nu})^{;\alpha}{}_c \tag{9b}$$

$$(h_a{}^e A_e^{\alpha\beta}{}_d)^{;\mu;\nu}{}_{b\ c} = (h_b{}^e A_{ed}^{\alpha\beta})^{;\mu;\nu}_{a\ c} \tag{9c}$$

$$(h_a{}^c B_c)^{;\mu}{}_b + (h_b{}^c B_c)^{;\mu}{}_a = 2(\partial/\partial x^\nu + \phi^{c;}{}_\nu\, \partial/\partial\phi^c)(h_a{}^d A_{db}^{\mu\nu}) \tag{9d}$$

$$(h_a{}^c B_c)^{;}{}_b - (h_b{}^c B_c)^{;}{}_a = \tfrac{1}{2}(\partial/\partial x^\nu + \phi^{c;}{}_\nu\, \partial/\partial\phi^c)[(h_a{}^c B_c)^{;\nu}{}_b - (h_b{}^c B_c)^{;\nu}{}_a] \tag{9e}$$

where

$$(h_a{}^d A_{db}^{\mu\nu})^{;\alpha}{}_c = \frac{\partial}{\partial\phi^{c;}{}_\alpha}(h_a{}^d A_{db}^{\mu\nu}) \qquad (h_a{}^c B_c)^{;}{}_b = \frac{\partial}{\partial\phi^b}(h_a{}^c B_c) \qquad \text{etc}$$

$$a, b, c, d, e = 1, 2, \ldots, n \qquad \mu, \nu, \alpha, \beta = 0, 1, 2, 3.$$

3. Theorem (equation (5))

Equation $(9d)$ can be transformed into

$$\left(\frac{\partial h_a{}^c}{\partial\phi^{b;}{}_\mu} + \frac{\partial h_b{}^c}{\partial\phi^{a;}{}_\mu}\right)B_c + h_a{}^c\frac{\partial B_c}{\partial\phi^{b;}{}_\mu} + h_b{}^c\frac{\partial B_c}{\partial\phi^{a;}{}_\mu} - 2\frac{\mathrm{d}}{\mathrm{d}x^\nu}(h_a{}^c A_{cb}^{\mu\nu})$$
$$+2\left(\frac{\partial}{\partial\phi^{d;}{}_\alpha}(h_a{}^c A_{cb})\right)\phi^{d;}{}_{\alpha\nu} = 0. \tag{10}$$

Using $(8a)$, $(8b)$, $(9a)$ and $(9b)$, the last term in equation (10) can be written as

$$2\left(\frac{\partial}{\partial\phi^{d;}{}_\alpha}(h_a{}^c A_{cb}^{\mu\nu})\right)\phi^{d;}{}_{\alpha\nu}$$
$$= \left(\frac{\partial h_a^c}{\partial\phi^{b;}{}_\mu} + \frac{\partial h_b^c}{\partial\phi^{a;}{}_\mu}\right)A_{cd}^{\alpha\nu}\phi^{d;}{}_{\alpha\nu} + h_a{}^c\frac{\partial A_{cd}^{\alpha\nu}}{\partial\phi^{b;}{}_\mu}\cdot\phi^{d;}{}_{\alpha\nu} + h_b{}^c\frac{\partial A_{cd}^{\alpha\nu}}{\partial\phi^{a;}{}_\mu}\cdot\phi^{d;}{}_{\alpha\nu} = 0. \tag{11}$$

By using equation (7) in equation (11), we have

$$2\left(\frac{\partial}{\partial\phi^{d;}{}_{\alpha}}(h_a{}^c A^{\mu\nu}_{cb})\right)\phi^{d;}{}_{\alpha\nu}$$

$$= -\left(\frac{\partial h_a{}^c}{\partial\phi^{b;}{}_{\mu}} + \frac{\partial h_b{}^c}{\partial\phi^{a;}{}_{\mu}}\right)B_c - h_a{}^c\frac{\partial B_c}{\partial\phi^{b;}{}_{\mu}} - h_b{}^c\frac{\partial B_c}{\partial\phi^{a;}{}_{\mu}}$$

$$- h_a{}^c A^{\alpha\nu}_{cd}\frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{b;}{}_{\mu}} - h_b{}^c A^{\alpha\nu}_{cd}\frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{a;}{}_{\mu}}.$$

The use of this result in (10) yields

$$2\frac{\mathrm{d}(h_a{}^c)}{\mathrm{d}x^{\nu}}\cdot A^{\mu\nu}_{cb} + 2h_a{}^c\frac{\mathrm{d}(A^{\mu\nu}_{cb})}{\mathrm{d}x^{\nu}} + h_a{}^c A^{\alpha\nu}_{cd}\frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{b;}{}_{\mu}} + h_b{}^c A^{\alpha\nu}_{cd}\frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{a;}{}_{\mu}} = 0. \tag{12}$$

On the other hand, from equations (8*d*) and (7), we get

$$2\,\mathrm{d}(A^{\mu\nu}_{ab})/\mathrm{d}x^{\nu} = -A^{\alpha\nu}_{ac}\,\partial\phi^{c;}{}_{\alpha\nu}/\partial\phi^{b;}{}_{\mu} - A^{\alpha\nu}_{bc}\,\partial\phi^{c;}{}_{\alpha\nu}/\partial\phi^{a;}{}_{\mu}. \tag{13}$$

By substituting (13) into (12), we have

$$2\frac{\mathrm{d}}{\mathrm{d}x^{\nu}}(h_a{}^c)\cdot A^{\mu\nu}_{cb} - \left(h_a{}^d\frac{\partial\phi^{c;}{}_{\alpha\nu}}{\partial\phi^{d;}{}_{\mu}} - \frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{a;}{}_{\mu}}\cdot h_d{}^c\right)A^{\alpha\nu}_{cb} = 0. \tag{14}$$

Equation (14) may be particularised by putting $A^{\mu\nu}_{ab} = g^{\mu\nu}A_{ab}$ (which reproduces equation (4)):

$$\left(2\frac{\mathrm{d}}{\mathrm{d}x^{\nu}}(h_a{}^c)\cdot g^{\mu\nu} - \left(h_a{}^d\cdot\frac{\partial\phi^{c;}{}_{\alpha\nu}}{\partial\phi^{d;}{}_{\mu}} - \frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{a;}{}_{\mu}}\cdot h_d{}^c\right)\cdot g^{\alpha\nu}\right)A_{cb} = 0. \tag{15}$$

Let us introduce a matrix $\tilde{A}$ such that

$$A_{cb}\tilde{A}^{bf} = \delta_c{}^f.$$

By multiplying equation (15) by this matrix $\tilde{A}$, we have

$$2\frac{\mathrm{d}}{\mathrm{d}x^{\nu}}(h_a{}^f)g^{\mu\nu} - \left(h_a{}^d\frac{\partial\phi^{f;}{}_{\alpha\nu}}{\partial\phi^{d;}{}_{\mu}} - \frac{\partial\phi^{d;}{}_{\alpha\nu}}{\partial\phi^{a;}{}_{\mu}}h_d^f\right)g^{\alpha\nu} = 0. \tag{16}$$

It is easy to see from equation (16) that

$$\mathrm{d}(h_a{}^a)/\mathrm{d}x^{\mu} = 0 \qquad a = 1, 2, \ldots, n \qquad \mu = 0, 1, 2, 3.$$

Thus statement (5) is proved.

A further generalisation of the previous result is achieved if

$$A^{\mu\nu}_{ab} = C^{\mu\nu}D_{ab}$$

where C and D are symmetric matrices, and D is non-singular.

Under such a hypothesis, we have

$$(\mathrm{d}/\mathrm{d}x^{\nu})(h_a{}^a)C^{\mu\nu} = 0 \qquad \nu, \mu = 0, 1, 2, 3.$$

If C is a diagonal matrix g, we have again equation (5).

4. Conclusions

Our result has an important and obvious application. The solution of the equivalence problem in its full generality implies obtaining h by solving the set of equations $(9a)-(9e)$ taking into account equations $(8a)-(8e)$ and (7). This, obviously, is an enormous task. However, if our result is used a great simplification is achieved since it allows a convenient ansatz for the form of h, as will be clarified by the following examples.

As a first example let us consider the equation for the damped vibration of a string

$$F_1 = (\partial^2\tilde{\phi}/\partial t^2) - C^2(\partial^2\tilde{\phi}/\partial x^2) - 2k(\partial\tilde{\phi}/\partial t) = 0$$

$$F_2 = (\partial^2\phi/\partial t^2) - C^2(\partial^2\phi/\partial x^2) + 2k(\partial\phi/\partial t) = 0.$$

The Lagrangian density is

$$\mathscr{L} = (\partial\tilde{\phi}/\partial t)(\partial\phi/\partial t) + k[\phi(\partial\tilde{\phi}/\partial t) - \tilde{\phi}(\partial\phi/\partial t)] - c^2(\partial\tilde{\phi}/\partial x)(\partial\phi/\partial x).$$

A particular solution for h is

$$h = \begin{bmatrix} 1 & A\,\mathrm{e}^{2kt} \\ A\,\mathrm{e}^{-2kt} & 1 \end{bmatrix}$$

where A is a numerical constant. The equivalent Lagrangian density is

$$\bar{\mathscr{L}} = \tfrac{1}{2}A\,\mathrm{e}^{2kt}[(\partial\phi/\partial t)^2 - c^2(\partial\phi/\partial x)^2] + \tfrac{1}{2}A\,\mathrm{e}^{-2kt}[(\partial\tilde{\phi}/\partial t)^2 - c^2(\partial\tilde{\phi}/\partial x)^2]$$
$$+ [(\partial\phi/\partial t)(\partial\tilde{\phi}/\partial t) - c^2(\partial\phi/\partial x)(\partial\tilde{\phi}/\partial x)] + k[\phi(\partial\tilde{\phi}/\partial t) - \tilde{\phi}(\partial\phi/\partial t)].$$

As a second example, consider the Lagrangian density

$$L = \tfrac{1}{2}(x+t)[(\partial\phi/\partial x)(\partial\tilde{\phi}/\partial x) - (\partial\phi/\partial t)(\partial\tilde{\phi}/\partial t)]$$
$$+ [(\partial\phi/\partial x) - (\partial\phi/\partial t)]\tilde{\phi} + \tfrac{1}{2}\phi[(\partial\tilde{\phi}/\partial x) - (\partial\tilde{\phi}/\partial t)] - \tfrac{1}{2}(\tilde{\phi})^2.$$

A particular solution for this problem is

$$h = \begin{bmatrix} 1 & 0 \\ B(x+t) & 1 \end{bmatrix}$$

where B is a numerical constant. The corresponding equivalent Lagrangian density is easily obtained:

$$\bar{L} = \tfrac{1}{2}(x+t)[(\partial\phi/\partial x)(\partial\tilde{\phi}/\partial x) - (\partial\phi/\partial t)(\partial\tilde{\phi}/\partial t)]$$
$$+ \tfrac{1}{4}B(x+t)^2[(\partial\tilde{\phi}/\partial x)^2 - (\partial\tilde{\phi}/\partial t)^2] - \tfrac{1}{2}\phi[(\partial\tilde{\phi}/\partial x) - (\partial\tilde{\phi}/\partial t)] - \tfrac{1}{2}(\tilde{\phi})^2.$$

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